

MORSE-NOVIKOV COHOMOLOGY OF CLOSED ONE-FORMS OF RANK 1

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ABSTRACT. We discuss the Morse-Novikov cohomology of a compact manifold, associated to a closed one-form whose free abelian group generated by its periods $\langle \int_\gamma \eta \mid [\gamma] \in \pi_1(M) \rangle$ is of rank 1, the focus being on locally conformally symplectic manifolds. In particular, we provide an explicit computation for the Inoue surface \mathcal{S}^0 .

Keywords: Morse-Novikov cohomology, Novikov ring, Novikov Betti numbers, locally conformally symplectic, Inoue surface \mathcal{S}^0 .

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1. INTRODUCTION

The Morse-Novikov cohomology of a manifold M refers to the cohomology of the complex of smooth real forms $\Omega^\bullet(M)$, with the differential operator perturbed with a closed one-form η , defined as follows

$$(1.1) \quad d_\eta := d - \eta \wedge \cdot$$

Indeed, the closedness of η implies $d_\eta^2 = 0$, whence d_η produces a cohomology, which we denote by $H_\eta^\bullet(M)$. Throughout this paper, we shall use the name Morse-Novikov for the cohomology $H_\eta^\bullet(M)$, although the name Lichnerowicz

cohomology is also used in the literature (see [BK], [HR]). Its study began with Novikov ([N1], [N2]) and was independently developed by Guedira and Lichnerowicz ([GL]).

The Morse-Novikov cohomology has more than one description. To begin with, let us look at the following exact sequence of sheafs:

$$(1.2) \quad 0 \rightarrow \text{Ker } d_\eta \xrightarrow{i} \Omega_M^0(\cdot) \xrightarrow{d_\eta} \Omega_M^1(\cdot) \xrightarrow{d_\eta} \Omega_M^2(\cdot) \xrightarrow{d_\eta} \dots$$

where we denote by $\Omega_M^k(\cdot)$ the sheaf of smooth real k -forms on M . In fact, the sequence above is an acyclic resolution for $\text{Ker } d_\eta$, as each $\Omega_M^i(\cdot)$ is soft (see Proposition 2.1.6 and Theorem 2.1.9 in [D]). Thus, by taking global sections in (1.2), we compute the cohomology groups of M with values in the sheaf $\text{Ker } d_\eta$, $H^i(M, \text{Ker } d_\eta)$. What we obtain is actually the Morse-Novikov cohomology.

The sheaf $\text{Ker } d_\eta$ has the property that there exists a covering $(U_i)_i$ of M , such that it is constant when restricted to each U_i . In order to see that, one simply takes a contractible covering $(U_i)_i$ for which $\eta = df_i|_{U_i}$, then by considering the map $g \mapsto e^{-f_i}g$, we get an isomorphism $\text{Ker } d_\eta(U_i) \simeq \mathbb{R}$.

Moreover, the covering $(U_i)_i$ and the isomorphisms above associate to $\text{Ker } d_\eta$ a line bundle L , which is trivial on this covering and whose transition maps are $g_{ij} = e^{f_i - f_j}$. It is immediate that (U_i, e^{-f_i}) defines a global nowhere vanishing section s of L^* , which is the dual of L and by means of s , L^* is isomorphic to the trivial bundle. We define a flat connection ∇ on L^* by $\nabla s = -\eta \otimes s$. Then $H_\eta^i(M)$ can also be computed as the cohomology of the following complex of forms with values in L^* :

$$(1.3) \quad 0 \rightarrow \Omega^0(M, L^*) \xrightarrow{\nabla} \Omega^1(M, L^*) \xrightarrow{\nabla} \Omega^2(M, L^*) \xrightarrow{\nabla} \dots$$

The Morse-Novikov cohomology is not a topological object in essence, however it can provide information about the closed one-form to which it is associated. For instance, it was shown in [LLMP] that if on a compact manifold M there exists a Riemannian metric g and a closed one-form η such that η is parallel with respect to g , then for any $i \geq 0$, $H_\eta^i(M) = 0$.

Some properties verified by the Morse-Novikov cohomology are summarized in the following:

Proposition 1.1: *Let M be a n -dimensional manifold and η a closed one-form. Then*

- (1) *if $\eta' = \eta + df$, for any $i \geq 0$, $H_{\eta'}^i(M) \simeq H_\eta^i(M)$ and the isomorphism is given by the map $[\alpha] \mapsto [e^{-f}\alpha]$.*
- (2) *([HR], [GL]) if η is not exact and M is connected and orientable, $H_\eta^0(M)$ and $H_\eta^n(M)$ vanish.*

- (3) ([BK]) *the Euler characteristic of the Morse-Novikov cohomology coincides with the Euler characteristic of the manifold, as a consequence of the Atiyah-Singer index theorem, which implies that the index of the elliptic complex $(\Omega^k(M), d_\eta)$ is independent of η .*

From now on, we shall assume throughout the paper that M is a compact manifold, unless specified. We denote by $\chi : \pi_1(M) \rightarrow \mathbb{R}$ the morphism of periods of η , namely $[\gamma] \mapsto \int_\gamma \eta$.

Definition 1.2: The *rank* of η is the rank of $\text{Im } \chi$ as a free abelian group.

The fundamental group of M is finitely presented, hence $\text{Im } \chi$ has finite rank and it is isomorphic to a free abelian group \mathbb{Z}^r . The following characterization was proven in [OV]:

Proposition 1.3: *The rank of a closed one-form η is the dimension of the smallest rational subspace $V \subset H^1(M, \mathbb{Q})$ such that $[\eta]$ lies in $V \otimes_{\mathbb{Q}} \mathbb{R}$.*

In other words, the rank of η is the maximum number of rationally independent periods.

Remark 1.4: If η is a closed one-form of rank 1, then $\text{Im } \chi = \alpha \cdot \mathbb{Z}$, where α is a real number.

Motivated by the natural setting that locally conformally symplectic manifolds provide for the Morse-Novikov cohomology, the aim of this note is to present some explicit examples and computations, of particular interest being the Inoue surface \mathcal{S}^0 .

The note is organized as follows. As we try to present this material as self-contained as possible, we give the necessary preliminaries in Section 2 and explain the tools we shall use in the sequel, namely a result of A. Pajitnov in [P], which relates the so-called *Novikov Betti numbers* to the Morse-Novikov cohomology and a twisted version of the Mayer-Vietoris sequence of S. Haller and T. Rybicki presented in [HR]. Section 3 is devoted to introducing locally conformally symplectic manifolds and to computing the Morse-Novikov cohomology of the Inoue surface \mathcal{S}^0 with respect to the closed one-form that F. Tricerri proves in [Tr] to be the Lee form of a locally conformally Kähler form.

2. PRELIMINARIES

We first give some definitions, in order to state later the results we are going to use.

Definition 2.1: Let $\Gamma \subset \mathbb{R}$ be a subgroup of \mathbb{R} . The *Novikov ring* associated to Γ is defined as the following ring of formal sums:

$$\text{Nov}(\Gamma) = \left\{ \sum_{i=1}^{\infty} n_i T^{\gamma_i} \mid n_i \in \mathbb{Z}, \gamma_i \in \Gamma, \lim_{i \rightarrow \infty} \gamma_i = -\infty \right\}$$

Remark 2.2: The ring $Nov(\Gamma)$ is a principal ideal domain (see [F, Lemma 1.15]).

In what follows, we consider Γ to be the group of periods of η . In this case, there is a $\mathbb{Z}[\pi_1(M)]$ -module structure of $Nov(\Gamma)$ described by $[\gamma] \cdot n = T^{-\int_\gamma \eta} \cdot n$, for any element n in $Nov(\Gamma)$. This further describes a $Nov(\Gamma)$ -local system on the manifold M , which we denote by $\widetilde{Nov(\Gamma)}$. We recall that if R is a commutative ring, there is a correspondence between representations of the fundamental group $\rho : \pi_1(M) \rightarrow \text{Aut}(R)$, $\mathbb{Z}[\pi_1(M)]$ -module structures on R and R -local systems. For more details, see Proposition 2.5.1 and Chapter 2 in [D].

Definition 2.3: For any $i \in \mathbb{N}$, the i -th *Novikov Betti number* is

$$b_i^{Nov}(M) := \text{rk}_{Nov(\Gamma)} H_i(M, \widetilde{Nov(\Gamma)}).$$

Remark 2.4: As M is compact, $H_i(M, \widetilde{Nov(\Gamma)})$ is a finitely generated module over a principal ideal domain and hence it decomposes as a direct sum of a free and a torsion part. We mean by $\text{rk}_{Nov(\Gamma)} H_i(M, \widetilde{Nov(\Gamma)})$ the rank of the free part of $H_i(M, \widetilde{Nov(\Gamma)})$.

The relation between the Morse-Novikov cohomology and the Novikov Betti numbers is given by the following result of A. Pajitnov, which we state in the form we shall need:

Theorem 2.5: ([P, Lemma 2]) *Let M be a manifold and η a closed one-form of rank 1, such that $\text{Im } \chi = \alpha\mathbb{Z}$, with e^α transcendental. Then for any $i \geq 0$:*

$$b_i^{Nov} = \dim_{\mathbb{R}} H_\eta^i(M).$$

The importance of the Novikov Betti numbers is of topological nature and refers to the extension of the classical Morse theory to closed one-forms developed by Novikov. His initial motivation was to find a relation between the zeros of a closed one-form η , of *Morse type* (namely, locally given by $\eta = df$, where f is a Morse function) and the topology of the manifold. The tool he created is a complex $(N_{\eta^*}^k, \delta_k)$, where $N_{\eta^*}^k$ is a free $Nov(\Gamma)$ -module, whose generators are in 1-1 correspondence with the zeros of index k of η . For the differentials δ_k , one needs to consider the cover $M_\eta \xrightarrow{\pi} M$ corresponding to the group $\text{Ker } \chi$, which is the minimal cover on which $\pi^*\eta$ is exact, and count down flow lines of a Smale vector field between critical points of consecutive index of the primitive of $\pi^*\eta$ (which turns out to be a Morse function). For more details regarding this construction, see [F], [L] and [Po].

The following result was stated by Novikov, [N2], but proven rigorously by F. Latour in [L] and M. Farber [F2].

Theorem 2.6: *Let M be a compact manifold, η a closed one-form of Morse type and $(N_{\eta*}, \delta_*)$ the Novikov complex associated to η . Then:*

$$H_i(N_{\eta*}, \delta_*) \simeq H_i(M, \widetilde{\text{Nov}(\Gamma)}).$$

Therefore, the relevance of the Novikov complex is that, like the Morse-Smale complex, it produces a topological result by using Morse theory. The complete proof of the theorem above can be found in [F, Chapter 3].

Remark 2.7: We notice that if η is a nowhere vanishing closed one-form, then the homology of $(N_{\eta*}, \delta_*)$ is 0 and by Theorem 2.6, all b_i^{Nov} vanish. If moreover, η satisfies the conditions in Theorem 2.5, the Morse-Novikov cohomology with respect to η also vanishes.

The second tool we shall use in this note, in order to compute the Morse-Novikov cohomology groups of \mathcal{S}^0 is the following version of the Mayer-Vietoris sequence:

Lemma 2.8: ([HR, Lemma 1.2]) *Let M be the union of two open sets U and V and θ a closed one-form. Then there exists a long exact sequence*

$$(2.1) \quad \cdots \rightarrow H_\theta^i(M) \xrightarrow{\alpha_*} H_{\theta|_U}^i(U) \oplus H_{\theta|_V}^i(V) \xrightarrow{\beta_*} H_{\theta|_{U \cap V}}^i(U \cap V) \xrightarrow{\delta} H_\theta^{i+1}(M) \rightarrow \cdots$$

where for some partition of unity $\{\lambda_U, \lambda_V\}$ subordinated to the covering $\{U, V\}$, the above morphisms are:

$$\begin{aligned} \delta([\sigma]) &= [d\lambda_U \wedge \sigma] = -[d\lambda_V \wedge \sigma], \\ \alpha(\sigma) &= (\sigma|_U, \sigma|_V), \\ \beta(\sigma, \tau) &= \sigma|_{U \cap V} - \tau|_{U \cap V}. \end{aligned}$$

Remark 2.9: In the case of the Inoue surface, we shall be interested in computing the Morse-Novikov cohomology of a closed one-form of rank 1, whose group of periods Γ is $\alpha\mathbb{Z}$ such that e^α is algebraic. The author in [P] gives an explicit computation, which covers this situation, as well. However, we use the Mayer-Vietoris sequence approach, since it is more direct.

3. LCS MANIFOLDS AND MORSE-NOVIKOV COHOMOLOGY OF THE INOUE SURFACE \mathcal{S}^0

Locally conformally symplectic manifolds (shortly LCS) are smooth real (necessarily even-dimensional) manifolds endowed with a nondegenerate two

form ω which satisfies the equality

$$(3.1) \quad d\omega = \theta \wedge \omega$$

for some closed one form θ , called the *Lee form*.

Equivalently, this means there exists a non-degenerate two-form ω , a covering of the manifold, $\{U_i\}_i$ and smooth functions f_i on U_i such that $e^{-f_i}\omega$ are symplectic, which literally explains their name.

The equality (3.1) rewrites as $d_\theta\omega = 0$, hence the problem of studying on an LCS manifold the Morse-Novikov cohomology associated to the Lee form of an LCS structure is natural.

Contact geometry is a source of examples of LCS manifolds. We adopt the following:

Definition 3.1: Let M be a manifold of odd dimension $2n + 1$. Then M is a *contact manifold* if there exists a one-form α such that $\alpha \wedge (d\alpha)^n$ is a volume form of M .

Definition 3.2: Let (M, α) be a contact manifold and $\varphi : M \rightarrow M$ a diffeomorphism. We call φ a *contactomorphism* if $\varphi^*\alpha = f \cdot \alpha$, where f is a positive function on M . In the case $f = 1$, φ is called a *strict contactomorphism*.

3.1. Mapping tori of contactomorphisms as LCS manifolds. We next describe a construction of compact LCS manifolds of rank 1 out of compact contact manifolds. The idea is to consider mapping tori of the latter by a contactomorphism.

Indeed, let (M, α) be a compact contact manifold and $\varphi : M \rightarrow M$ a contactomorphism. We define

$$\overline{M}_\varphi := M \times [0, 1] / (x, 0) \sim (\varphi(x), 1).$$

We shall denote by $[m, t]$ a point of \overline{M}_φ , which is the equivalence class of (m, t) in $M \times [0, 1]$. Then \overline{M}_φ has a natural structure of fiber bundle over S^1 with fiber M , given by $\pi : \overline{M}_\varphi \rightarrow S^1$, $\pi([m, t]) = e^{2\pi it}$. Here, S^1 is seen as the interval $[0, 1]$ with identified endpoints.

Let h be a smooth bump function, which is 0 near 0 and 1 near 1. On $M \times [0, 1]$, define the one-form $\tilde{\alpha}$ by:

$$\tilde{\alpha} := \alpha(hf + (1 - h)).$$

Then $\tilde{\alpha}$ descends to a one-form α_1 on \overline{M}_φ which has the property that restricted to any fiber M_t is a contact form.

We denote by ϑ the volume form of the circle of length 1. The two-form $d\alpha_1 - \pi^*\vartheta \wedge \alpha_1$ is closed with respect to $d_{\pi^*\vartheta}$, but it is possible to have degeneracy points. Since \overline{M}_φ is compact, we may choose a large constant $K \gg 0$ such that

$$\omega := d\alpha_1 - K \cdot \pi^*\vartheta \wedge \alpha_1$$

is nondegenerate. Moreover, we can choose K in such a way that e^K is not algebraic. Then ω defines an LCS form with the Lee form $\theta = K\pi^*\vartheta$.

Clearly, the Lee form $\pi^*\vartheta$ is integral, and hence θ is a closed one-form of rank 1, with the group of periods $K\mathbb{Z}$.

By the choice of K , the one-form θ is in the situation described by Theorem 2.5, therefore the dimensions of the Morse-Novikov cohomology groups equal the Novikov Betti numbers of θ . Since θ is nowhere vanishing, we obtain:

Proposition 3.3: *Let \overline{M}_φ and θ as above. Then the Novikov Betti numbers vanish and the Morse-Novikov cohomology of θ is 0.*

Remark 3.4: Let us modify θ with an exact form df such that $\theta_1 := \theta + df$ has at least one zero. By point (3) in Proposition 1.1, we get that $H_\theta(\overline{M}_\varphi) = H_{\theta_1}(\overline{M}_\varphi) = 0$, hence the vanishing of the Morse-Novikov cohomology may occur also for forms which have zeros. In particular, the converse of the result in [LLMP] mentioned in Section 1 is not true. Namely, if Morse-Novikov cohomology vanishes, the one-form with respect to which it is considered may not be parallel, since a parallel one-form has no zeros.

Remark 3.5: The construction above describes a large class of LCS manifolds. The particular case when φ is the identity gives the product $M \times S^1$. In the case when φ is a strict contactomorphism, there is no need of choosing the bump function h , since α defines a global form on \overline{M}_φ . Then a straightforward computation shows that $d\alpha - \pi^*\vartheta \wedge \alpha$ is nondegenerate, hence the constant K may be chosen to be 1. This situation is described in [B] and many examples are given in [BM].

3.2. The Inoue surface S^0 . On the Inoue surfaces of type S^0 we consider a closed-one form of rank one, for which e^α is algebraic. We present the explicit description of the Morse-Novikov cohomology groups.

3.2.1. Description of the LCS manifold S^0 . In [I], M. Inoue introduced three types of complex compact surfaces, which are traditionally referred to as the Inoue surfaces S^0 , S^+ and S^- . In [Tr], Tricerri endowed the Inoue surfaces S^0 , S^+ and some subclasses of S^- with locally conformally Kähler metrics, in particular, by forgetting the complex structure, with locally conformally symplectic structures.

We are interested in the LCS structure on S^0 and we compute its corresponding Morse-Novikov cohomology. For this purpose, we review the construction of S^0 and insist on its description as mapping torus of the 3-dimensional torus \mathbb{T}^3 .

Let A be a matrix from $\mathrm{SL}_3(\mathbb{Z})$ with one real eigenvalue $\alpha > 1$ and two complex eigenvalues β and $\overline{\beta}$. We denote by $(a_1, a_2, a_3)^t$ a real eigenvector of α and by $(b_1, b_2, b_3)^t$ a complex eigenvector of β . Let G be the group of

affine transformations of $\mathbb{C} \times \mathbb{H}$ generated by the transformations:

$$\begin{aligned}(z, w) &\mapsto (\beta z, \alpha w), \\ (z, w) &\mapsto (z + b_i, w + a_i).\end{aligned}$$

for all $i = 1, 2, 3$, where \mathbb{H} stands for the Poincaré half-plane.

As a complex manifold, \mathcal{S}^0 is $(\mathbb{C} \times \mathbb{H})/G$.

We now explain its structure as a mapping torus. Denote by \mathbb{T}^3 the standard 3-dimensional torus, namely $\mathbb{T}^3 = \mathbb{R}^3 / \langle f_1, f_2, f_3 \rangle$, where f_1 (resp. f_2, f_3) is the translation with $(1, 0, 0)$ (resp. $(0, 1, 0), (0, 0, 1)$).

Let $\mathbf{x} := (x, y, z)^t$, and consider the automorphism Φ of \mathbb{R}^3 with matrix A^t in the canonical basis. It clearly descends to an automorphism φ of \mathbb{T}^3 , since A^t belongs to $\mathrm{SL}_3(\mathbb{Z})$. We define the manifold

$$\mathbb{T}^3 \times_{\varphi} \mathbb{R}^+ := (\mathbb{T}^3 \times \mathbb{R}^+) / (\widehat{\mathbf{x}}, t) \sim (\varphi(\widehat{\mathbf{x}}), \alpha t)$$

which has the structure of a compact fiber bundle over S^1 by considering

$$p : \mathbb{T}^3 \times_{\varphi} \mathbb{R}^+ \rightarrow S^1, \quad [(\widehat{\mathbf{x}}, t)] \mapsto e^{2\pi i \log_{\alpha} t}$$

Here we denote by $[\cdot]$ the equivalence class with respect to \sim .

In order to write explicitly a diffeomorphism between $\mathbb{T}^3 \times_{\varphi} \mathbb{R}^+$ and \mathcal{S}^0 , let

$$B := \begin{pmatrix} \mathrm{Re} b_1 & \mathrm{Re} b_2 & \mathrm{Re} b_3 \\ \mathrm{Im} b_1 & \mathrm{Im} b_2 & \mathrm{Im} b_3 \\ a_1 & a_2 & a_3 \end{pmatrix}$$

Now the requested diffeomorphism acts as:

$$[\widehat{\mathbf{x}}, t] \mapsto [[\widehat{B \cdot \mathbf{x}}, t]],$$

where $x + iy$ and $z + it$ are coordinates on $\mathbb{C} \times \mathbb{H}$ and $[[x + iy, z + it]]$ denotes the equivalence class of $(x + iy, z + it)$. It is straightforward to check this map is well defined and indeed an isomorphism.

The LCK structure given by Tricerri in [Tr] is given as a G -invariant globally conformally Kähler structure on $\mathbb{C} \times \mathbb{H}$ and in the coordinates (z, w) , the expressions for the metric and the Lee form, respectively, are:

$$\begin{aligned}g &= -i \frac{dw \otimes d\bar{w}}{w_2^2} + w_2 dz \otimes d\bar{z} \\ \theta &= \frac{dw_2}{w_2},\end{aligned}$$

where $w_2 = \mathrm{Im}(w)$. For our description as fiber bundle and coordinates (x, y, z, t) , the Lee form θ is $\frac{dt}{t}$.

As previously, we denote by ϑ the volume form of the circle of length 1. Obviously, the de Rham cohomology class $[\vartheta]$ is in $H^1(S^1, \mathbb{Z})$ and implicitly $[p^*\vartheta]$ belongs to $H^1(\mathcal{S}^0, \mathbb{Z})$, and hence the rank is 1. Moreover, a simple computation shows that

$$\theta = \ln \alpha \cdot p^*\vartheta.$$

So we are not in the situation depicted by Theorem 2.5, since $e^{\ln \alpha} = \alpha$ is an algebraic integer.

3.2.2. Explicit computation of the Morse-Novikov cohomology. To compute by hand the Morse-Novikov cohomology groups of \mathcal{S}^0 with the twisted Mayer-Vietoris sequence, we first choose the open sets U_1 and U_2 which cover the circle:

$$U_1 := \{e^{2\pi it} \mid t \in (0, 1)\}, \quad U_2 := \{e^{2\pi it} \mid t \in (\frac{1}{2}, \frac{3}{2})\},$$

and take as open sets $U := p^{-1}(U_1)$ and $V := p^{-1}(U_2)$, representing a covering of \mathcal{S}^0 . The sets U and V are the trivializations of \mathcal{S}^0 as fiber bundle over S^1 . Therefore, we have

$$\begin{aligned} \varphi_{U_1} : U &\longrightarrow U_1 \times \mathbb{T}^3, & [w, t] &\mapsto (e^{2\pi it}, w), \quad t \in (1, \alpha), \\ \varphi_{U_2} : V &\longrightarrow U_2 \times \mathbb{T}^3, & [w, t] &\mapsto (e^{2\pi it}, w), \quad t \in (\alpha^{\frac{1}{2}}, \alpha^{\frac{3}{2}}). \end{aligned}$$

Since $U_1 \cap U_2$ is disconnected, the transition maps $g_{U_1 U_2} := \varphi_{U_1} \circ \varphi_{U_2}^{-1}$ are given by:

$$\begin{aligned} g_{U_1 U_2} : U_1 \cap U_2 \times \mathbb{T}^3 &\rightarrow U_1 \cap U_2 \times \mathbb{T}^3, \\ g_{U_1 U_2}(m, \mathbf{x}) &= \begin{cases} (m, \widehat{\mathbf{x}}), & \text{if } m = e^{2\pi it}, \text{ with } t \in (\frac{1}{2}, 1) \\ (m, (\widehat{A^t})^{-1} \cdot \mathbf{x}), & \text{if } m = e^{2\pi it}, \text{ with } t \in (1, \frac{3}{2}) \end{cases} \end{aligned}$$

As θ is not exact, we already know that $H_\theta^0(\mathcal{S}^0)$ and $H_\theta^4(\mathcal{S}^0)$ vanish (see [HR]). Concerning the other Morse-Novikov cohomology groups, we prove the following result:

Theorem 3.6: *On \mathcal{S}^0 , for the Lee form θ given by Tricerri, $H_\theta^1(\mathcal{S}^0)$ vanishes, $H_\theta^2(\mathcal{S}^0) \simeq \mathbb{R}$ and $H_\theta^3(\mathcal{S}^0) \simeq \mathbb{R}$.*

Proof. The proof is algebraic and the key is to explicitly write the morphism β_* .

From now on we denote by W_1 and W_2 the two connected components of $U_1 \cap U_2$, namely

$$W_1 = \{e^{2\pi it} \mid t \in (\frac{1}{2}, 1)\}, \quad W_2 = \{e^{2\pi it} \mid t \in (1, \frac{3}{2})\}.$$

Consider the functions $f : U_1 \rightarrow (0, 1)$, $f(e^{2\pi it}) = t$ and $g : U_2 \rightarrow (\frac{1}{2}, \frac{3}{2})$, $g(e^{2\pi it}) = t$. Then on U_1 , $\vartheta = df$ and on U_2 , $\vartheta = dg$. Moreover, we observe that on W_1 , f and g coincide and on W_2 , $g = f + 1$. Therefore, $\theta = \ln \alpha \cdot dp^* f$ on U and $\theta = \ln \alpha \cdot dp^* g$ on V , hence θ is exact on these two open sets.

We have the following diagram:

$$\begin{array}{ccc} H_{\theta|_U}^0(U) \oplus H_{\theta|_V}^0(V) & \xrightarrow{\beta_*} & H_{\theta|_{U \cap V}}^0(U \cap V) \\ \Phi \downarrow & & \downarrow \Psi \\ \mathbb{R}^2 & \xrightarrow{\gamma} & \mathbb{R}^2 \end{array}$$

where Φ and Ψ are the isomorphisms defined as

$$\begin{aligned} \Phi([\sigma], [\eta]) &= (e^{-\ln \alpha p^* f} \sigma, e^{-\ln \alpha p^* g} \eta), \\ \Psi([\omega]) &= (e^{-\ln \alpha p^* f} \omega|_{p^{-1}(W_1)}, e^{-\ln \alpha p^* f} \omega|_{p^{-1}(W_2)}), \end{aligned}$$

and γ makes the diagram commutative, $\gamma(a, b) = (a - b, a - \alpha b)$.

As $\alpha \neq 1$, γ is an isomorphism, and hence β_* is an isomorphism, too. Consequently, the connecting morphism $\delta : H_{\theta|_{U \cap V}}^0(U \cap V) \rightarrow H_\theta^1(\mathcal{S}^0)$ is injective and we can start the Mayer-Vietoris from $H_\theta^1(\mathcal{S}^0)$:

$$\begin{aligned} 0 \rightarrow H_\theta^1(\mathcal{S}^0) \rightarrow H_{\theta|_U}^1(U) \oplus H_{\theta|_V}^1(V) \rightarrow H_{\theta|_{U \cap V}}^1(U \cap V) \rightarrow \\ \rightarrow \cdots \rightarrow H_{\theta|_{U \cap V}}^3(U \cap V) \rightarrow 0 \end{aligned}$$

We look now at the other morphisms β_* linking cohomology groups of degree $i \geq 1$.

$$\begin{array}{ccc} H_{\theta|_U}^i(U) \oplus H_{\theta|_V}^i(V) & \xrightarrow{\beta_*} & H_{\theta|_{U \cap V}}^i(U \cap V) \\ \Phi \downarrow & & \downarrow \Psi \\ H_{dR}^i(\mathbb{T}^3) \oplus H_{dR}^i(\mathbb{T}^3) & \xrightarrow{\gamma} & H_{dR}^i(\mathbb{T}^3) \oplus H_{dR}^i(\mathbb{T}^3) \end{array}$$

Using the fact that θ is exact when restricted to U and V , the isomorphism Φ is obtained by the following composition of isomorphisms:

$$H_{\theta|_U}^i(U) \xrightarrow{f_1} H_{dR}^i(U) \xrightarrow{f_2} H_{dR}^i(U_1 \times \mathbb{T}^3) \xrightarrow{f_3} H_{dR}^i(\mathbb{T}^3),$$

where $f_1([\sigma]) = [e^{-f} \sigma]$, $f_2([\eta]) = [(\varphi_{U_1})_* \eta]$, $f_3([\omega]) = [i^* \omega]$ and $i : \mathbb{T}^3 \rightarrow U_1 \times \mathbb{T}^3$ is defined as $i(t) = (m, t)$, for some point m in U_1 .

The same holds for V , the only difference being that $f'_1 : H_{\theta|_V}^i(V) \rightarrow H_{dR}^i(V)$ is given by $[\sigma] \mapsto [e^{-g} \sigma]$ and $f'_2 : H_{dR}^i(V) \rightarrow H_{dR}^i(U_2 \times \mathbb{T}^3)$ is given by $[\eta] \mapsto [(\varphi_{U_2})_* \eta]$. Thus:

$$\Phi = f_3 \circ f_2 \circ f_1 \oplus f'_3 \circ f'_2 \circ f'_1.$$

As for Ψ , there is a similar sequence:

$$H_{\theta|_{U \cap V}}^i(U \cap V) \xrightarrow{g_1} H_{dR}^i(U \cap V) \xrightarrow{g_2} H_{dR}^i(U \cap V \times \mathbb{T}^3) \xrightarrow{g_3} H_{dR}^i(\mathbb{T}^3) \oplus H_{dR}^i(\mathbb{T}^3).$$

Here, the isomorphisms g_1 , g_2 and g_3 are given by $[\sigma] \mapsto [e^{-f} \sigma]$, $[\eta] \mapsto [(\varphi_U)_* \eta]$ and $[\omega] \mapsto (i_1^*[\omega|_{W_1}], i_2^*[\omega|_{W_2}])$, where $i_1 : \mathbb{T}^3 \rightarrow W_1 \times \mathbb{T}^3$ denotes the

injection $t \mapsto (m, t)$ for some m in W_1 and $i_2 : \mathbb{T}^3 \rightarrow W_2 \times \mathbb{T}^3$, $i_2(t) = (n, t)$ for some point n in W_2 . We define $\Psi = g_3 \circ g_2 \circ g_1$.

A straightforward computation shows that $\gamma = \Phi^{-1} \circ \beta_* \circ \Psi$ is given by:

$$([a], [b]) \mapsto ([a - b], [a - \alpha \cdot i_2^*((g_{U_1 U_2})|_{W_2})_* \pi^* b]),$$

where $\pi : V \times \mathbb{T}^3 \rightarrow \mathbb{T}^3$ is the projection on the second factor.

We investigate now the map $i_2^*((g_{U_1 U_2})|_{W_2})_* \pi^* : H_{dR}^i(\mathbb{T}^3) \rightarrow H_{dR}^i(\mathbb{T}^3)$ for $i = 1, 2, 3$. It is an easy observation that

$$i_2^*((g_{U_1 U_2})|_{W_2})_* \pi^* = (\pi \circ (g_{U_1 U_2})|_{W_2} \circ i_2)_*.$$

Since $\pi \circ (g_{U_1 U_2})|_{W_2} \circ i_2 : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ is given by the matrix $(A^t)^{-1}$, the map induced in homology, $(\pi \circ (g_{U_1 U_2})|_{W_2} \circ i_2)_* : H_1(\mathbb{T}^3) \rightarrow H_1(\mathbb{T}^3)$ has the matrix $(A^t)^{-1}$ in the canonical basis. Therefore, the matrix of the map induced by the pushforward $(\pi \circ (g_{U_1 U_2})|_{W_2} \circ i_2)_* : H_{dR}^1(\mathbb{T}^3) \rightarrow H_{dR}^1(\mathbb{T}^3)$ in the canonical basis $\{[dx], [dy], [dz]\}$ is $((A^t)^{-1})^t = A$.

As a consequence, we obtain that the matrix of $\gamma : H_{dR}^1(\mathbb{T}^3) \oplus H_{dR}^1(\mathbb{T}^3) \rightarrow H_{dR}^1(\mathbb{T}^3) \oplus H_{dR}^1(\mathbb{T}^3)$ is the following:

$$\left[\begin{array}{c|c} I_3 & -I_3 \\ \hline I_3 & -\alpha \cdot A \end{array} \right]$$

By performing a transformation which keeps the rank constant, namely adding the first three columns to the last three, we obtain that the aforementioned matrix has the same rank as:

$$\left[\begin{array}{c|c} I_3 & O_3 \\ \hline I_3 & I_3 - \alpha \cdot A \end{array} \right]$$

Moreover, this further implies that the rank is controlled by the block $I_3 - \alpha \cdot A$, which would be a nonsingular matrix if and only if $\frac{1}{\alpha}$ were an eigenvalue of A , which is not the case. Hence, γ and implicitly β_* is an isomorphism, whence from the Mayer-Vietoris sequence, $H_\theta^1(\mathcal{S}^0)$ has to vanish.

Since we already know the matrix of $(\pi \circ (g_{U_1 U_2})|_{W_2} \circ i_2)_* : H_{dR}^1(\mathbb{T}^3) \rightarrow H_{dR}^1(\mathbb{T}^3)$ is A in the basis $\{[dx], [dy], [dz]\}$, we can easily compute the matrix of $(\pi \circ (g_{U_1 U_2})|_{W_2} \circ i_2)_* : H_{dR}^2(\mathbb{T}^3) \rightarrow H_{dR}^2(\mathbb{T}^3)$ in the basis $\{[dy \wedge dz], [dz \wedge dx], [dx \wedge dy]\}$ to be $(A^*)^t$. Therefore, the matrix of $\gamma : H_{dR}^2(\mathbb{T}^3) \oplus H_{dR}^2(\mathbb{T}^3) \rightarrow H_{dR}^2(\mathbb{T}^3) \oplus H_{dR}^2(\mathbb{T}^3)$ is:

$$\left[\begin{array}{c|c} I_3 & -I_3 \\ \hline I_3 & -\alpha \cdot (A^*)^t \end{array} \right]$$

which by the same arguments as above has the same rank as:

$$\left[\begin{array}{c|c} I_3 & O_3 \\ \hline I_3 & I_3 - \alpha \cdot (A^*)^t \end{array} \right]$$

Since $A^* = A^{-1}$ (because A lives in $\mathrm{SL}_3(\mathbb{Z})$) and a matrix and its transpose have the same eigenvalues, $(A^*)^t$ has the same eigenvalues as A^{-1} , thus $\frac{1}{\alpha}$ is one of them. Therefore, the rank of the block $I_3 - \alpha \cdot (A^*)^t$ is 2, because $\frac{1}{\alpha}$ is an eigenvalue of $(A^*)^t$ of multiplicity 1. We infer that the matrix of $\gamma : H_{dR}^2(\mathbb{T}^3) \oplus H_{dR}^2(\mathbb{T}^3) \rightarrow H_{dR}^2(\mathbb{T}^3) \oplus H_{dR}^2(\mathbb{T}^3)$ has rank 5, forcing $\mathrm{Ker} \gamma$ to be 1-dimensional and from the Mayer-Vietoris sequence, we obtain $H_\theta^2(\mathcal{S}^0) \simeq \mathbb{R}$.

For the final case, when $i = 3$, it is straightforward that $(\pi \circ (g_{U_1 U_2})|_{W_2} \circ i_2)_* : H_{dR}^3(\mathbb{T}^3) \rightarrow H_{dR}^3(\mathbb{T}^3)$ is given by the multiplication with the determinant of the matrix of $(\pi \circ (g_{U_1 U_2})|_{W_2} \circ i_2)_* : H_{dR}^1(\mathbb{T}^3) \rightarrow H_{dR}^1(\mathbb{T}^3)$. In this case, the determinant is 1, hence we get that $\gamma : H_{dR}^3(\mathbb{T}^3) \oplus H_{dR}^3(\mathbb{T}^3) \rightarrow H_{dR}^3(\mathbb{T}^3) \oplus H_{dR}^3(\mathbb{T}^3)$ is given by the 2×2 -matrix:

$$\begin{bmatrix} 1 & -1 \\ 1 & -\alpha \end{bmatrix}$$

and thus it defines an isomorphism. By the Mayer-Vietoris sequence, we obtain:

$$\dim_{\mathbb{R}} H_\theta^3(\mathcal{S}^0) = 6 - \dim_{\mathbb{R}} \mathrm{Im}(\beta_* : H_{\theta|_U}^2(U) \oplus H_{\theta|_V}^2(V) \rightarrow H_{\theta|_{U \cap V}}^2(U \cap V)) = 1$$

In conclusion, $H_\theta^3(\mathcal{S}^0) \simeq \mathbb{R}$, $H_\theta^2(\mathcal{S}^0) \simeq \mathbb{R}$ and the rest of the Morse-Novikov cohomology groups vanish. \blacksquare

We now find generators for $H_\theta^2(\mathcal{S}^0)$ and $H_\theta^3(\mathcal{S}^0)$.

Denote by

$$\Omega := -i \left(\frac{dw \wedge d\bar{w}}{w_2^2} + w_2 dz \wedge d\bar{z} \right)$$

the two-form on $\mathbb{C} \times \mathbb{H}$, in the coordinates (z, w) , which descends to an LCS form ω on \mathcal{S}^0 . Notice that $\Omega_1 := -i \frac{dw \wedge d\bar{w}}{w_2^2}$ and $\Omega_2 := -i w_2 dz \wedge d\bar{z}$ are two-forms which are invariant with respect to the factorization group G . They descend to \mathcal{S}^0 to two forms which we shall denote by ω_1 and ω_2 and we have $\omega = \omega_1 + \omega_2$. Tricerri showed that ω is an LCK form and it is the fundamental two-form of the metric induced by $g = -i \frac{dw \otimes d\bar{w}}{w_2^2} + w_2 dz \otimes d\bar{z}$ on \mathcal{S}^0 , which we shall denote by g_1 . Then we have the following:

Proposition 3.7: *Let ω be the above defined LCS form of \mathcal{S}^0 and $\theta = \frac{dw_2}{w_2}$ its Lee form, as in Theorem 3.6. Then:*

$$\begin{aligned} H_\theta^2(\mathcal{S}^0) &= \mathbb{R}[\omega] \\ H_\theta^3(\mathcal{S}^0) &= \mathbb{R}[\theta \wedge \omega]. \end{aligned}$$

Before proving these equalities, we define the notion of *twisted laplacian*. Namely, by extending the metric g_1 to the space of k -forms $\Omega^k(\mathcal{S}^0)$, we consider the Hodge star operator $*$: $\Omega^k(\mathcal{S}^0) \rightarrow \Omega^{4-k}(\mathcal{S}^0)$, given by $u \wedge *v = g_1(u, v)d\text{vol}$. Note that the real dimension of \mathcal{S}^0 is 4. Then the following operators depending on θ can be defined (they indeed make sense on any manifold M endowed with a closed one-form θ , although we shall treat specifically the case of \mathcal{S}^0):

$$\begin{aligned}\delta_\theta : \Omega^{k+1}(\mathcal{S}^0) &\rightarrow \Omega^k(\mathcal{S}^0), & \delta_\theta &= - * d_{-\theta} * \\ \Delta_\theta : \Omega^k(\mathcal{S}^0) &\rightarrow \Omega^k(\mathcal{S}^0), & \Delta_\theta &= \delta_\theta d_\theta + d_\theta \delta_\theta\end{aligned}$$

Remark 3.8: δ_θ is the adjoint of d_θ with respect to the inner product on $\Omega^k(\mathcal{S}^0)$ given by $\langle \eta, \varphi \rangle = \int_{\mathcal{S}^0} \eta \wedge * \varphi$. Observe that δ_θ and Δ_θ are perturbations of the usual operators codifferential and laplacian, which are recovered by replacing θ with 0. The motivation for introducing the operators twisted with θ is to develop Hodge theory in the context of working with d_θ instead of d . They were first considered in [Val] in the context of locally conformally Kähler manifolds and later in [GL] in the LCS setting.

The following analogue of Hodge decomposition holds:

Theorem 3.9: ([GL]) *Let M be a compact manifold, θ a closed one-form, δ_θ and Δ_θ defined as above. Then we have an orthogonal decomposition:*

$$(3.2) \quad \Omega^k(M) = \mathcal{H}_\theta^k(M) \oplus d_\theta \Omega^{k-1}(M) \oplus \delta_\theta \Omega^{k+1}(M)$$

where $\mathcal{H}_\theta^k(M) = \{\eta \in \Omega^k(M) \mid \Delta_\theta \eta = 0\}$. Moreover,

$$H_\theta^k(M) \simeq \mathcal{H}_\theta^k(M).$$

Thus, we observe that important properties of the Hodge-de-Rham theory for the operator d are shared by the same theory applied to d_θ .

We now give the

Proof of Proposition 3.7. Since we proved in Theorem 3.6 that H_θ^2 and H_θ^3 are isomorphic to \mathbb{R} , it is enough to show that ω and $\theta \wedge \omega$ are d_θ -closed, but not d_θ -exact.

We shall prove that with respect to the Hodge decomposition (3.2), ω has the harmonic and the d_θ -exact parts non-vanishing. Indeed, a straightforward computation shows that $\Omega_1 = d \frac{dw_2}{w_2} \frac{-dw_1}{w_2}$. Since $\frac{-dw_1}{w_2}$ is G -invariant and descends to a one-form η on \mathcal{S}^0 , we have $w_1 = d_\theta \eta$. As ω is the fundamental two-form of the metric g_1 , which is hermitian with respect to the complex structure of \mathcal{S}^0 induced from the standard one on $\mathbb{C} \times \mathbb{H}$, an easy linear algebra computation (see [GH, p. 31]) shows that the Riemannian volume form $d\text{vol}$ equals $\frac{\omega^2}{2!}$. In the general case of complex dimension n , the volume form $d\text{vol}$ is $\frac{\omega^n}{n!}$. This further implies that $*\omega_2 = \omega_1$. Consequently, $d_{-\theta} * \omega_2 = d_{-\theta} \omega_1 = d\omega_1 + \theta \wedge \omega_1$. However, $d\Omega_1 = 0$ and $\frac{dw_2}{w_2} \wedge \Omega_1 = 0$, hence

$d\omega_1 = 0$ and $\theta \wedge \omega_1 = 0$, implying that ω_2 is δ_θ -closed. Still, one can show that Ω_2 is $d_{\frac{d\omega_2}{\omega_2}}$ -closed, therefore ω_2 also is d_θ -closed. So ω_2 is harmonic with respect to Δ_θ . Thus, $\omega = \omega_1 + \omega_2$ is the Hodge decomposition of ω . We proved in this way that ω is not d_θ -exact and moreover, $[\omega] = [\omega_2]$ defines a non-vanishing cohomology class in $H_\theta^2(M)$. But $H_\theta^2(M) \simeq \mathbb{R}$, therefore $H_\theta^2(M) = \mathbb{R}[\omega] = \mathbb{R}[\omega_2]$.

As for $H_\theta^3(M)$, we first notice that $\theta \wedge \omega$ is d_θ -closed. Indeed, $d_\theta(\theta \wedge \omega) = d(d\omega) - \theta \wedge \theta \wedge \omega = 0$. In [G], it was shown that $\Delta_\theta(\theta \wedge \omega) = 0$, whence we obtain, as in the case of ω , that $\theta \wedge \omega$ is not d_θ -exact. This means that we found a generator for $H_\theta^3(M)$, namely $H_\theta^3(M) = \mathbb{R}[\theta \wedge \omega]$. ■

Remark 3.10: We notice that the alternate sum of the dimensions of the Morse-Novikov cohomology $H_\theta^i(\mathcal{S}^0)$ groups is 0, which equals indeed the Euler characteristic of \mathcal{S}^0 .

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